CHAPTER
8

THE STEADY MAGNETIC FIELD

At this point the concept of a field should be a familiar one. Since we first accepted the experimental law of forces existing between two point charges and defined electric field intensity as the force per unit charge on a test charge in the presence of a second charge, we have discussed numerous fields. These fields possess no real physical basis, for physical measurements must always be in terms of the forces on the charges in the detection equipment. Those charges which are the source cause measurable forces to be exerted on other charges, which we may think of as detector charges. The fact that we attribute a field to the source charges and then determine the effect of this field on the detector charges amounts merely to a division of the basic problem into two parts for convenience.

We shall begin our study of the magnetic field with a definition of the magnetic field itself and show how it arsies from a current distribution. The effect of this field on other currents, or the second half of the physical problem, will be discussed in the following chapter. As we did with the electric field, we shall confine our initial discussion to free-space conditions, and the effect of material media will also be saved for discussion in the following chapter.

The relation of the steady magnetic field to its source is more complicated than is the relation of the electrostatic field to its source. We shall find it necessary to accept several laws temporarily on faith alone, relegating their proof to the (rather difficult) final section in this chapter. This section may well be omitted when studying magnetic fields for the first time. It is included to make acceptance of the laws a little easier; the proof of the laws does exist and is available for the disbelievers or the more advanced student.

8.1 BIOT-SAVART LAW

The source of the steady magnetic field may be a permanent magnet, an electric field changing linearly with time, or a direct current. We shall largely ignore the permanent magnet and save the time-varying electric field for a later discussion. Our present relationships will concern the magnetic field produced by a differential dc element in free space.

We may think of this differential current element as a vanishingly small section of a current-carrying filamentary conductor, where a filamentary conductor is the limiting case of a cylindrical conductor of circular cross section as the radius approaches zero. We assume a current I flowing in a different vector length of the filament $d\mathbf{L}$. The law of Biot-Savart[1] then states that at any point P the magnitude of the magnetic field intensity produced by the differential element is proportional to the product of the current, the magnitude of the differential length, and the sine of the angle lying between the filament and a line connecting the filament to the point P at which the field is desired; also, the magnitude of the magnetic field intensity is inversely proportional to the square of the distance from the differential element to the point P. The direction of the magnetic field intensity is normal to the plane containing the differential filament and the line drawn from the filament to the point P. Of the two possible normals, that one is to be chosen which is in the direction of progress of a right-handed screw turned from $d\mathbf{L}$ through the smaller angle to the line from the filament to P. Using rationalized mks units, the constant of proportionality is $1/4\pi$.

The *Biot-Savart law*, described above in some 150 words, may be written concisely using vector notation as

$$d\mathbf{H} = \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \frac{I d\mathbf{L} \times \mathbf{R}}{4\pi R^3} \tag{1}$$

The units of the *magnetic field intensity* $\mathbf{H}$ are evidently amperes per meter (A/m). The geometry is illustrated in Fig. 8.1. Subscripts may be used to indicate the point to which each of the quantities in (1) refers. If we locate the current element at point 1 and describe the point P at which the field is to be determined as point 2, then

[1] Biot and Savart were colleagues of Ampère, and all three were professors of physics at the Collège de France at one time or another. The Biot-Savart law was proposed in 1820.

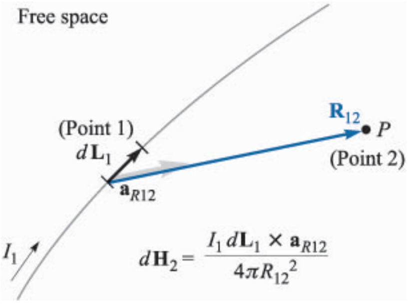

FIGURE 8.1
The law of Biot-Savart expresses the magnetic field intensity $d\mathbf{H}_2$ produced by a differential current element $I_1 d\mathbf{L}_1$. The direction of $d\mathbf{H}_2$ is into the page.

$$d\mathbf{H}_2 = \frac{I_1 d\mathbf{L}_1 \times \mathbf{a}_{R12}}{4\pi R_{12}^2} \tag{2}$$

The law of Biot-Savart is sometimes called *Ampère's law for the current element*, but we shall retain the former name because of possible confusion with Ampère's circuital law, to be discussed later.

In some aspects, the Biot-Savart law is reminiscent of Coulomb's law when that law is written for a differential element of charge,

$$d\mathbf{E}_2 = \frac{dQ_1 \mathbf{a}_{R12}}{4\pi\epsilon_0 R_{12}^2}$$

Both show an inverse-square-law dependence on distance, and both show a linear relationship between source and field. The chief difference appears in the direction of the field.

It is impossible to check experimentally the law of Biot-Savart as expressed by (1) or (2) because the differential current element cannot be isolated. We have restricted our attention to direct currents only, so the charge density is not a function of time. The continuity equation in Sec. 5.2, Eq. (5),

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$$

therefore shows that

$$\nabla \cdot \mathbf{J} = 0$$

or upon applying the divergence theorem,

$$\oint_s \mathbf{J} \cdot d\mathbf{S} = 0$$

The total current crossing any closed surface is zero, and this condition may be satisfied only by assuming a current flow around a closed path. It is this current flowing in a closed circuit which must be our experimental source, not the differential element.

It follows that only the integral form of the Biot-Savart law can be verified experimentally,

$$\mathbf{H} = \oint \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} \tag{3}$$

Equation (1) or (2), of course leads directly to the integral form (3), but other differential expressions also yield the same integral formulation. Any term may be added to (1) whose integral around a closed path is zero. That is, any conservative field could be added to (1). The gradient of any scalar field always yields a conservative field, and we could therefore add a term ∇G to (1), where G is a general scalar field, without changing (3) in the slightest. This qualification on (1) or (2) is mentioned to show that if we later ask some foolish questions, not subject to any experimental check, concerning the force exerted by one *differential* current element on another, we should expect foolish answers.

The Biot-Savart law may also be expressed in terms of distributed sources, such as current density **J** and *surface current density* **K**. Surface current flows in a sheet of vanishingly small thickness, and the current density **J**, measured in amperes per square meter, is therefore infinite. Surface current density, however, is measured in amperes per meter width and designated by **K**. If the surface current density is uniform, the total current I in any width b is

$$I = Kb$$

where we have assumed that the width b is measured perpendicularly to the direction in which the current is flowing. The geometry is illustrated by Fig. 8.2. For a nonuniform surface current density, integration is necessary:

$$I = \int K dN \tag{4}$$

where dN is a differential element of the path *across* which the current is flowing. Thus the differential current element $I\,d\mathbf{L}$, where $d\mathbf{L}$ is in the direction of the

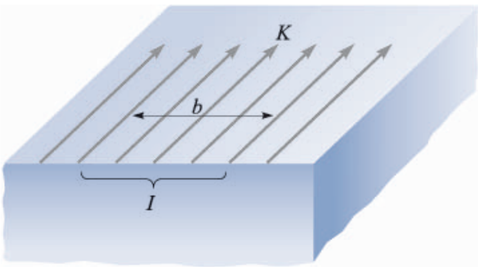

FIGURE 8.2
The total current I within a transverse width b, in which there is a *uniform* surface current density K, is Kb.

current, may be expressed in terms of surface current density **K** or current density **J**,

$$I\,d\mathbf{L} = \mathbf{K}\,dS = \mathbf{J}\,dv \tag{5}$$

and alternate forms of the Biot-Savart law obtained,

$$\mathbf{H} = \int_s \frac{\mathbf{K} \times \mathbf{a}_R dS}{4\pi R^2} \tag{6}$$

and

$$\mathbf{H} = \int_{\text{vol}} \frac{\mathbf{J} \times \mathbf{a}_R dv}{4\pi R^2} \tag{7}$$

We may illustrate the application of the Biot-Savart law by considering an infinitely long straight filament. We shall apply (2) first and then integrate. This, of course, is the same as using the integral form (3) in the first place.[2]

Referring to Fig. 8.3, we should recognize the symmetry of this field. No variation with z or with ϕ can exist. Point 2, at which we shall determine the field, is therefore chosen in the $z = 0$ plane. The field point **r** is therefore $r = \rho\mathbf{a}_\rho$. The source point $\mathbf{r}'$ is given by $\mathbf{r}' = z'\mathbf{a}_z$, and therefore

$$\mathbf{R}_{12} = \mathbf{r} - \mathbf{r}' = \rho\mathbf{a}_\rho - z'\mathbf{a}_z$$

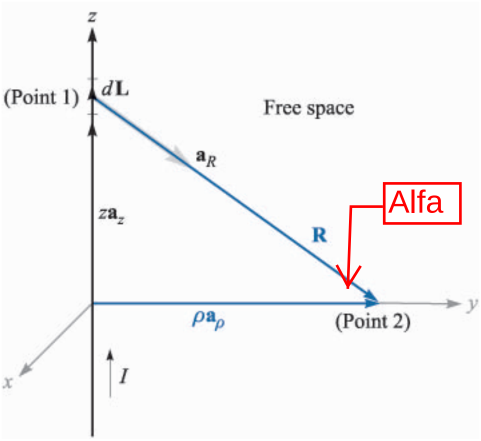

FIGURE 8.3
An infinitely long straight filament carrying a direct current I. The field at point 2 is $\mathbf{H} = (I/2\pi\rho)\mathbf{a}_\phi$.

[2] The closed path for the current may be considered to include a return filament parallel to the first filament and infinitely far removed. An outer coaxial conductor of infinite radius is another theoretical possibility. Practically, the problem is an impossible one, but we should realize that our answer will be quite accurate near a very long straight wire having a distant return path for the current.

so that

$$\mathbf{a}_{R12} = \frac{\rho \mathbf{a}_\rho - z' \mathbf{a}_z}{\sqrt{\rho^2 + z'^2}}$$

We take $d\mathbf{L} = dz' \mathbf{a}_z$ and (2) becomes

$$d\mathbf{H}_2 = \frac{I\, dz' \mathbf{a}_z \times (\rho \mathbf{a}_\rho - z' \mathbf{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}$$

Since the current is directed toward increasing values of z', the limits are $-\infty$ and ∞ on the integral, and we have

$$\mathbf{H}_2 = \int_{-\infty}^{\infty} \frac{I\, dz' \mathbf{a}_z \times (\rho \mathbf{a}_\rho - z' \mathbf{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}$$
$$= \frac{I}{4\pi} \int_{-\infty}^{\infty} \frac{\rho\, dz' \mathbf{a}_\phi}{(\rho^2 + z'^2)^{3/2}}$$

At this point the unit vector $\mathbf{a}_\phi$, under the integral sign should be investigated, for it is not always a constant, as are the unit vectors of the cartesian coordinate system. A vector is constant when its magnitude and direction are both constant. The unit vector certainly has constant magnitude, but its direction may change. Here $\mathbf{a}_\phi$ changes with the coordinate ϕ but not with ρ or z. Fortunately, the integration here is with respect to z', and $\mathbf{a}_\phi$ is a constant and may be removed from under the integral sign,

$$\mathbf{H}_2 = \frac{I \rho \mathbf{a}_\phi}{4\pi} \int_{-\infty}^{\infty} \frac{dz'}{(\rho^2 + z'^2)^{3/2}}$$
$$= \frac{I \rho \mathbf{a}_\phi}{4\pi} \left. \frac{z'}{\rho^2 \sqrt{\rho^2 + z'^2}} \right|_{-\infty}^{\infty}$$

=sin (alfa)

and

$$\mathbf{H}_2 = \frac{I}{2\pi\rho} \mathbf{a}_\phi \qquad (8)$$

The magnitude of the field is not a function of ϕ or z and it varies inversely as the distance from the filament. The direction of the magnetic-field-intensity vector is circumferential. The streamlines are therefore circles about the filament, and the field may be mapped in cross section as in Fig. 8.4.

The separation of the streamlines is proportional to the radius, or inversely proportional to the magnitude of **H**. To be specific, the streamlines have been drawn with curvilinear squares in mind. As yet we have no name for the family of lines[3] which are perpendicular to these circular streamlines, but the spacing of

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[3] If you can't wait, see Sec. 8.6

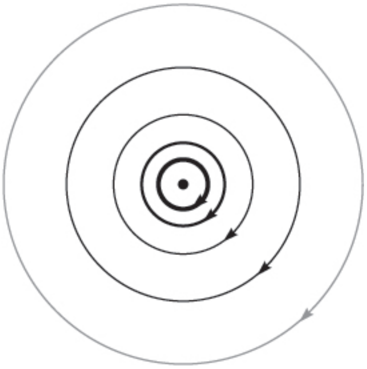

FIGURE 8.4
The streamlines of the magnetic field intensity about an infinitely long straight filament carrying a direct current I. The direction of I is into the page.

Later Comparison

the streamlines has been adjusted so that the addition of this second set of lines will produce an array of curvilinear squares.

A comparison of Fig. 8.4 with the map of the *electric* field about an infinite line *charge* shows that the streamlines of the magnetic field correspond exactly to the equipotentials of the electric field, and the unnamed (and undrawn) perpendicular family of lines in the magnetic field corresponds to the streamlines of the electric field. This correspondence is not an accident, but there are several other concepts which must be mastered before the analogy between electric and magnetic fields can be explored more thoroughly.

Using the Biot-Savart law to find **H** is in many respects similar to the use of Coulomb's law to find **E**. Each requires the determination of a moderately complicated integrand containing vector quantities, followed by an integration. When we were concerned with Coulomb's law we solved a number of examples, including the fields of the point charge, line charge, and sheet of charge. The law of Biot-Savart can be used to solve analogous problems in magnetic fields, and some of these problems now appear as exercises at the end of the chapter rather than as examples here.

One useful result is the field of the finite-length current element, shown in Fig. 8.5. It turns out (see Prob. 8 at the end of the chapter) that **H** is most easily expressed in terms of the angles α_1 and α_2, as identified in the figure. The result is

$$\mathbf{H} = \frac{I}{4\pi\rho}(\sin\alpha_2 - \sin\alpha_1)\mathbf{a}_\phi \qquad (9)$$

If one or both ends are below point 2, then α_1, or both α_1 and α_2, are negative.

Equation (9) may be used to find the magnetic field intensity caused by current filaments arranged as a sequence of straight line segments.

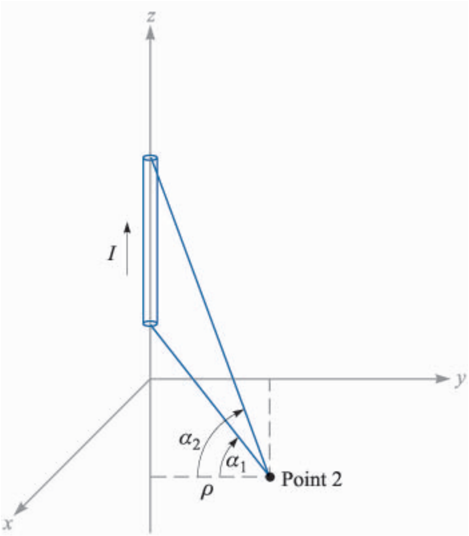

FIGURE 8.5
The magnetic field intensity caused by a finite-length current filament on the z axis is $(I/4\pi\rho)(\sin\alpha_2 - \sin\alpha_1)\mathbf{a}_\phi$.

Example 8.1

As a numerical example illustrating the use of (9), let us determine **H** at $P_2(0.4, 0.3, 0)$ in the field of an 8-A filamentary current directed inward from infinity to the origin on the positive x axis, and then outward to infinity along the y axis. This arrangement is shown in Figure 8.6.

Solution. We first consider the semi-infinite current on the x axis, identifying the two angles, $\alpha_{1x} = -90^\circ$ and $\alpha_{2x} = \tan^{-1}(0.4/0.3) = 53.1^\circ$. The radial distance ρ is measured from the x axis, and we have $\rho_x = 0.3$. Thus, this contribution to $\mathbf{H}_2$ is

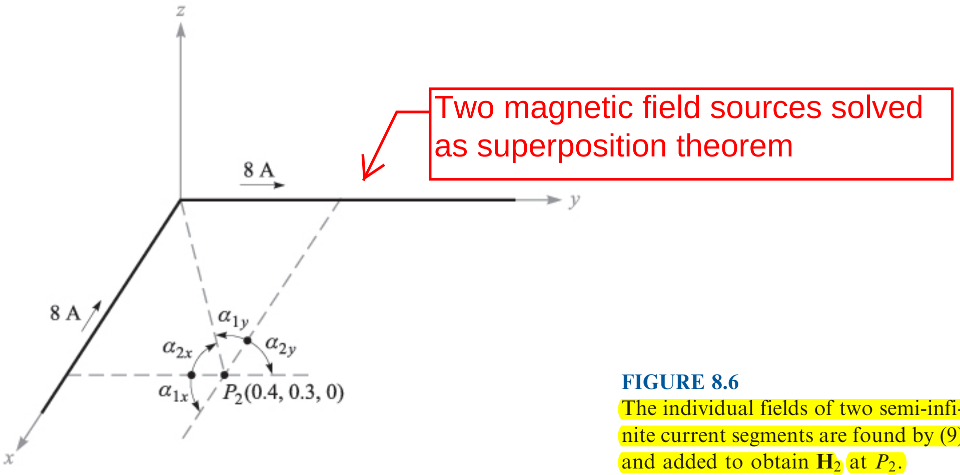

FIGURE 8.6
The individual fields of two semi-infinite current segments are found by (9) and added to obtain $\mathbf{H}_2$ at P_2.

$$\mathbf{H}_{2(x)} = \frac{8}{4\pi(0.3)}(\sin 53.1^\circ + 1)\mathbf{a}_\phi = \frac{2}{0.3\pi}(1.8)\mathbf{a}_\phi = \frac{12}{\pi}\mathbf{a}_\phi$$

The unit vector $\mathbf{a}_\phi$, must also be referred to the x axis. We see that it becomes $-\mathbf{a}_z$. Therefore,

$$\mathbf{H}_{2(x)} = -\frac{12}{\pi}\mathbf{a}_z \quad \text{A/m}$$

For the current on the y axis, we have $\alpha_{1y} = -\tan^{-1}(0.3/0.4) = -36.9^\circ$, $\alpha_{2y} = 90^\circ$, and $\rho_y = 0.4$. It follows that

$$\mathbf{H}_{2(y)} = \frac{8}{4\pi(0.4)}(1 + \sin 36.9^\circ)(-\mathbf{a}_z) = -\frac{8}{\pi}\mathbf{a}_z \quad \text{A/m}$$

Adding these results, we have

$$\mathbf{H}_2 = \mathbf{H}_{2(x)} + \mathbf{H}_{2(y)} = -\frac{20}{\pi}\mathbf{a}_z = -6.37\mathbf{a}_z \quad \text{A/m}$$

✔ **D8.1.** Given the following values for P_1, P_2, and $I_1\Delta_1$, calculate $\Delta\mathbf{H}_2$: (*a*) $P_1(0, 0, 2)$, $P_2(4, 2, 0)$, $2\pi\mathbf{a}_z\mu\text{A}\cdot\text{m}$; (*b*) $P_1(0, 2, 0)$, $P_2(4, 2, 0)$, $2\pi\mathbf{a}_z\mu\text{A}\cdot\text{m}$; (*c*) $P_1(1, 2, 3)$, $P_2(-3, -1, 2)$, $2\pi(-\mathbf{a}_x + \mathbf{a}_y + 2\mathbf{a}_z)\mu\text{A}\cdot\text{m}$.

Ans. $-8.51\mathbf{a}_x + 17.01\mathbf{a}_y$ nA/m; $16\mathbf{a}_y$ nA/m; $3.77\mathbf{a}_x - 6.79\mathbf{a}_y + 5.28\mathbf{a}_z$ nA/m

✔ **D8.2.** A current filament carrying 15 A in the $\mathbf{a}_z$ direction lies along the entire z axis. Find $\mathbf{H}$ in cartesian coordinates at: (*a*) $P_A(\sqrt{20}, 0, 4)$; (*b*) $P_B(2, -4, 4)$.

Ans. $0.534\mathbf{a}_y$ A/m; $0.477\mathbf{a}_x + 0.239\mathbf{a}_y$ A/m

8.2 AMPÈRE'S CIRCUITAL LAW

After solving a number of simple electrostatic problems with Coulomb's law, we found that the same problems could be solved much more easily by using Gauss's law whenever a high degree of symmetry was present. Again, an analogous procedure exists in magnetic fields. Here, the law that helps us solve problems more easily is known as *Ampère's circuital*[4] *law*, sometimes called Ampère's work law. This law may be derived from the Biot-Savart law, and the derivation is accomplished in Sec. 8.7. For the present we might agree to accept Ampère's circuital law temporarily as another law capable of experimental proof. As is the case with Gauss's law, its use will also require careful consideration of the symmetry of the problem to determine which variables and components are present.

Ampère's circuital law states that the line integral of **H** about any *closed* path is exactly equal to the direct current enclosed by that path,

[4] The preferred pronunciation puts the accent on "circ-."

$$\oint \mathbf{H} \cdot d\mathbf{L} = I \tag{10}$$

We define positive current as flowing in the direction of advance of a right-handed screw turned in the direction in which the closed path is traversed.

Referring to Fig. 8.7, which shows a circular wire carrying a direct current I, the line integral of **H** about the closed paths lettered a and b results in an answer of I; the integral about the closed path c which passes through the conductor gives an answer less than I and is exactly that portion of the total current which is enclosed by the path c. Although paths a and b give the same answer, the integrands are, of course, different. The line integral directs us to multiply the component of **H** in the direction of the path by a small increment of path length at one point of the path, move along the path to the next incremental length, and repeat the process, continuing until the path is completely traversed. Since **H** will generally vary from point to point, and since paths a and b are not alike, the contributions to the integral made by, say, each micrometer of path length are quite different. Only the final answers are the same.

We should also consider exactly what is meant by the expression "current enclosed by the path." Suppose we solder a circuit together after passing the conductor once through a rubber band, which we shall use to represent the closed path. Some strange and formidable paths can be constructed by twisting and knotting the rubber band, but if neither the rubber band nor the conducting circuit is broken, the current enclosed by the path is that carried by the conductor. Now let us replace the rubber band by a circular ring of spring steel across which is stretched a rubber sheet. The steel loop forms the closed path, and the current-carrying conductor must pierce the rubber sheet if the current is to be enclosed by the path. Again, we may twist the steel loop, and we may also deform the rubber sheet by pushing our fist into it or folding it in any way we wish. A single current-carrying conductor still pierces the sheet once, and this is the true measure of the current enclosed by the path. If we should thread the conductor once through the sheet from front to back and once from back to front, the total current enclosed by the path is the algebraic sum, which is zero.

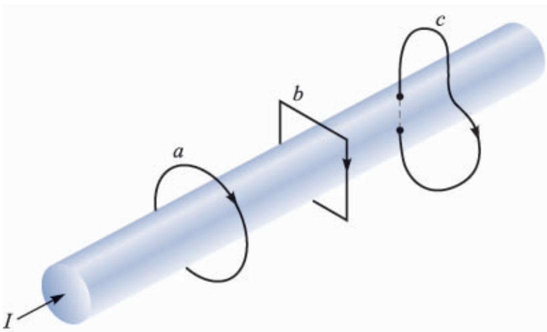

FIGURE 8.7
A conductor has a total current I. The line integral of **H** about the closed paths a and b is equal to I, and the integral around path c is less than I, since the entire current is not enclosed by the path.

In more general language, given a closed path, we recognize this path as the perimeter of an infinite number of surfaces (not closed surfaces). Any current-carrying conductor enclosed by the path must pass through every one of these surfaces once. Certainly some of the surfaces may be chosen in such a way that the conductor pierces them twice in one direction and once in the other direction, but the algebraic total current is still the same.

We shall find that the nature of the closed path is usually extremely simple and can be drawn on a plane. The simplest surface is, then, that portion of the plane enclosed by the path. We need merely find the total current passing through this region of the plane.

The application of Gauss's law involves finding the total charge enclosed by a closed surface; the application of Ampère's circuital law involves finding the total current enclosed by a closed path.

Let us again find the magnetic field intensity produced by an infinitely long filament carrying a current I. The filament lies on the z axis in free space (as in Fig. 8.3), and the current flows in the direction given by $\mathbf{a}_z$. Symmetry inspection comes first, showing that there is no variation with z or ϕ. Next we determine which components of $\mathbf{H}$ are present by using the Biot-Savart law. Without specifically using the cross product, we may say that the direction of $d\mathbf{H}$ is perpendicular to the plane conaining $d\mathbf{L}$ and $\mathbf{R}$ and therefore is in the direction of $\mathbf{a}_\phi$. Hence the only component of $\mathbf{H}$ is H_ϕ, and it is a function only of ρ.

We therefore choose a path to any section of which $\mathbf{H}$ is either perpendicular or tangential and along which H is constant. The first requirement (perpendicularity or tangency) allows us to replace the dot product of Ampère's circuital law with the product of the scalar magnitudes, except along that portion of the path where $\mathbf{H}$ is normal to the path and the dot product is zero; the second requirement (constancy) then permits us to remove the magnetic field intensity from the integral sign. The integration required is usually trivial and consists of finding the length of that portion of the path to which $\mathbf{H}$ is parallel.

In our example the path must be a circle of radius ρ and Ampère's circuital law becomes

$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{2\pi} H_\phi \rho \, d\phi = H_\phi \rho \int_0^{2\pi} d\phi = H_\phi 2\pi\rho = I$$

or

$$H_\phi = \frac{I}{2\pi\rho}$$

This law make H easy to derived into the circular coordinates.

as before.

As a second example of the application of Ampère's circuital law, consider an infinitely long coaxial transmission line carrying a uniformly distributed total current I in the center conductor and $-I$ in the outer conductor. The line is shown in Fig. 8.8*a*. Symmetry shows that H is not a function of ϕ or z. In order to determine the components present, we may use the results of the previous

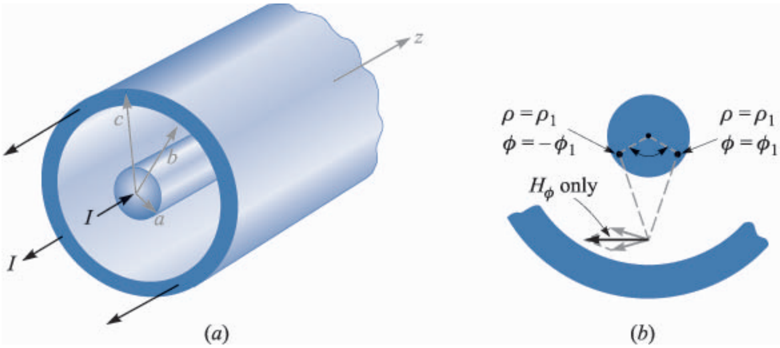

FIGURE 8.8
(*a*) Cross section of a coaxial cable carrying a uniformly distributed current I in the inner conductor and $-I$ in the outer conductor. The magnetic field at any point is most easily determined by applying Ampère's circuital law about a circular path. (*b*) Current filaments at $\rho = \rho_1$, $\phi = \pm\phi_1$, produces $\mathbf{H}_\rho$ components which cancel. For the total field, $\mathbf{H} = \mathbf{H}_\phi \mathbf{a}_\phi$.

example by considering the solid conductors as being composed of a large number of filaments. No filament has a z component of $\mathbf{H}$. Furthermore, the H_ρ component at $\phi = 0^\circ$, produced by one filament located at $\rho = \rho_1$, $\phi = \phi_1$, is canceled by the H_ρ component produced by a symmetrically located filament at $\rho = \rho_1$, $\phi = -\phi_1$. This symmetry is illustrated by Fig. 8.8*b*. Again we find only an H_ϕ component which varies with ρ.

A circular path of radius ρ, where ρ is larger than the radius of the inner conductor but less than the inner radius of the outer conductor, then leads immediately to

$$H_\phi = \frac{I}{2\pi\rho} \qquad (a < \rho < b)$$

If we choose ρ smaller than the radius of the inner conductor, the current enclosed is

$$I_{encl} = I\frac{\rho^2}{a^2}$$

and

$$2\pi\rho H_\phi = I\frac{\rho^2}{a^2}$$

or

$$H_\phi = \frac{I\rho}{2\pi a^2} \qquad (\rho < a)$$

If the radius ρ is larger than the outer radius of the outer conductor, no current is enclosed and

$$H_\phi = 0 \qquad (\rho > c)$$

Finally, if the path lies within the outer conductor, we have

$$2\pi\rho H_\phi = I - I\left(\frac{\rho^2 - b^2}{c^2 - b^2}\right)$$

pi .rho^2-pi .b^2= part of area up to point rho.
c^2-b^2= total area of outer conductor

$$H_\phi = \frac{I}{2\pi\rho}\frac{c^2 - \rho^2}{c^2 - b^2} \qquad (b < \rho < c)$$

The magnetic-field-strength variation with radius is shown in Fig. 8.9 for a coaxial cable in which $b = 3a$, $c = 4a$. It should be noted that the magnetic field intensity **H** is continuous at all the conductor boundaries. In other words, a slight increase in the radius of the closed path does not result in the enclosure of a tremendously different current. The value of H_ϕ shows no sudden jumps.

The external field is zero. This, we see, results from equal positive and negative currents enclosed by the path. Each produces an external field of magnitude $I/2\pi\rho$, but complete cancellation occurs. This is another example of "shielding"; such a coaxial cable carrying large currents would not produce any noticeable effect in an adjacent circuit.

As a final example, let us consider a sheet of current flowing in the positive y direction and located in the $z = 0$ plane. We may think of the return current as equally divided between two distant sheets on either side of the sheet we are considering. A sheet of uniform surface current density $\mathbf{K} = K_y\mathbf{a}_y$ is shown in Fig. 8.10. **H** cannot vary with x or y. If the sheet is subdivided into a number of filaments, it is evident that no filament can produce an H_y component. Moreover, the Biot-Savart law shows that the contributions to H_z produced by a symmetrically located pair of filaments cancel. Thus, H_z is zero also; only an H_x component is present. We therefore choose the path 1-1′-2′-2-1 composed of straight-line segments which are either parallel or perpendicular to H_x. Ampère's circuital law gives

$$H_{x1}L + H_{x2}(-L) = K_yL$$

or

$$H_{x1} - H_{x2} = K_y$$

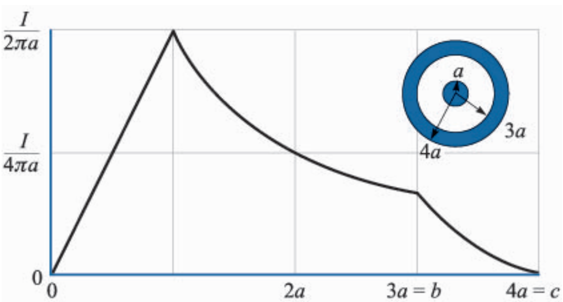

FIGURE 8.9
The magnetic field intensity as a function of radius in an infinitely long coaxial transmission line with the dimensions shown.

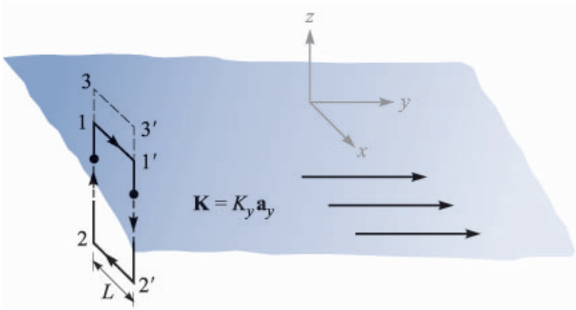

FIGURE 8.10
A uniform sheet of surface current $\mathbf{K} = K_y \mathbf{a}_y$ in the $z = 0$ plane. $\mathbf{H}$ may be found by applying Ampère's circuital law about the paths 1-1′-2′-2-1 and 3-3′-2′-2-3.

If the path 3-3′-2′-2-3 is now chosen, the same current is enclosed, and

$$H_{x3} - H_{x2} = K_y$$

and therefore

$$H_{x3} = H_{x1}$$

It follows that H_x is the same for all positive z. Similarly, H_x is the same for all negative z. Because of the symmetry, then, the magnetic field intensity on one side of the current sheet is the negative of that on the other. Above the sheet,

$$H_x = \tfrac{1}{2} K_y \quad (z > 0)$$

while below it

$$H_x = -\tfrac{1}{2} K_y \quad (z < 0)$$

Letting $\mathbf{a}_N$ be a unit vector normal (outward) to the current sheet, the result may be written in a form correct for all z as

$$\boxed{\mathbf{H} = \tfrac{1}{2} \mathbf{K} \times \mathbf{a}_N} \tag{11}$$

If a second sheet of current flowing in the opposite direction, $\mathbf{K} = -K_y \mathbf{a}_y$, is placed at $z = h$, (11) shows that the field in the region between the current sheets is

$$\boxed{\mathbf{H} = \mathbf{K} \times \mathbf{a}_N \quad (0 < z < h)} \tag{12}$$

and is zero elsewhere,

$$\boxed{\mathbf{H} = 0 \quad (z < 0, z > h)} \tag{13}$$

The most difficult part of the application of Ampère's circuital law is the determination of the components of the field which are present. The surest method is the logical application of the Biot-Savart law and a knowledge of the magnetic fields of simple form.

Problem 13 at the end of this chapter outlines the steps involved in applying Ampère's circuital law to an infinitely long solenoid of radius a and uniform current density $K_a \mathbf{a}_\phi$, as shown in Fig. 8.11a. For reference, the result is

$$\mathbf{H} = K_a \mathbf{a}_z \qquad (\rho < a) \tag{14a}$$

$$\mathbf{H} = 0 \qquad (\rho > a) \tag{14b}$$

If the solenoid has a finite length d and consists of N closely wound turns of a filament that carries a current I (Fig. 8.11b), then the field at points well within the solenoid is given closely by

$$\mathbf{H} = \frac{NI}{d}\mathbf{a}_z \qquad \text{(well within the solenoid)} \tag{15}$$

The approximation is useful it if is not applied closer than two radii to the open ends, nor closer to the solenoid surface than twice the separation between turns.

For the toroids shown in Fig. 8.12, it can be shown that the magnetic field intensity for the ideal case, Fig. 8.12a, is

$$\mathbf{H} = K_a \frac{\rho_0 - a}{\rho} \mathbf{a}_\phi \qquad \text{(inside toroid)} \tag{16a}$$

$$\mathbf{H} = 0 \qquad \text{(outside)} \tag{16b}$$

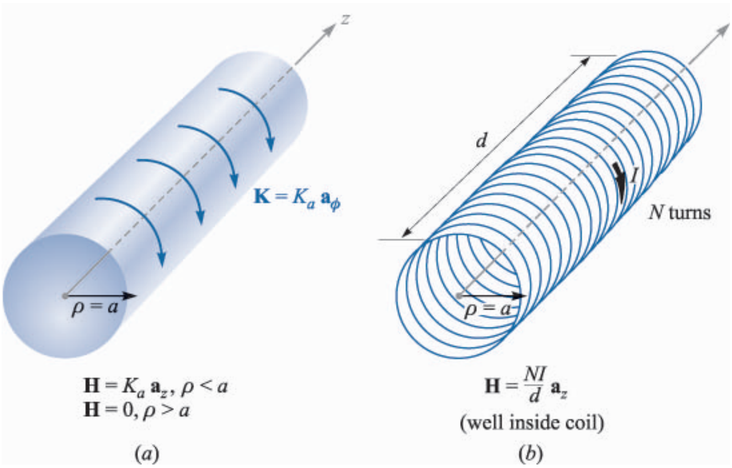

FIGURE 8.11
(a) An ideal solenoid of infinite length with a circular current sheet $\mathbf{K} = K_a \mathbf{a}_\phi$. ($b$) An N-turn solenoid of finite length d.

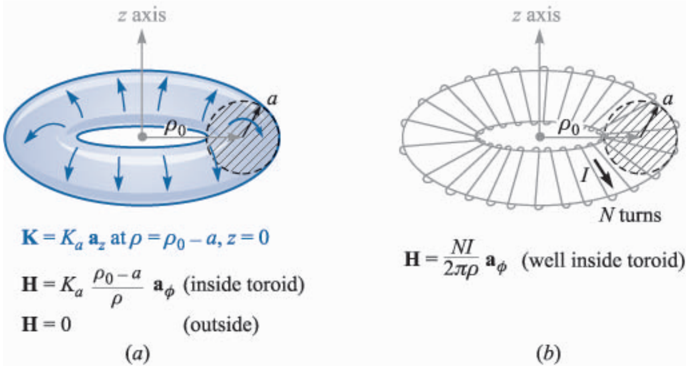

FIGURE 8.12
(*a*) An ideal toroid carrying a surface current **K** in the direction shown. (*b*) An *N*-turn toroid carrying a filamentary current *I*.

For the *N*-turn toroid of Figure 8.12*b*, we have the good approximations,

$$\mathbf{H} = \frac{NI}{2\pi\rho}\mathbf{a}_\phi \quad \text{(inside toroid)} \tag{17a}$$

$$\mathbf{H} = 0 \quad \text{(outside)} \tag{17b}$$

as long as we consider points removed from the toroidal surface by several times the separation between turns.

Toroids having rectangular cross sections are also treated quite readily, as you can see for yourself by trying Prob. 14.

Accurate formulas for solenoids, toroids, and coils of other shapes are available in Sec. 2 of the "Standard Handbook for Electrical Engineers" (see Suggested References for Chap. 5).

✔ **D8.3.** Express the value of **H** in cartesian components at $P(0, 0.2, 0)$ in the field of: (*a*) a current filament, 2.5 A in the $\mathbf{a}_z$ direction at $x = 0.1$, $y = 0.3$; (*b*) a coax, centered on the z axis, with $a = 0.3$, $b = 0.5$, $c = 0.6$, $I = 2.5$ A in $\mathbf{a}_z$ direction in center conductor; (*c*) three current sheets, $2.7\mathbf{a}_x$ A/m at $y = 0.1$, $-1.4\mathbf{a}_x$ A/m at $y = 0.15$, and $-1.3\mathbf{a}_x$ A/m at $y = 0.25$.

Ans. $1.989\mathbf{a}_x - 1.989\mathbf{a}_y$ A/m; $-0.884\mathbf{a}_x$ A/m; $1.300\mathbf{a}_z$ A/m

8.3 CURL

We completed our study of Gauss's law by applying it to a differential volume element and were led to the concept of divergence. We now apply Ampère's circuital law to the perimeter of a differential surface element and discuss the third and last of the special derivatives of vector analysis, the curl. Our immediate objective is to obtain the point form of Ampère's circuital law.

Again we shall choose cartesian coordinates, and an incremental closed path of sides Δx and Δy is selected (Fig. 8.13). We assume that some current, as yet unspecified, produces a reference value for **H** at the *center* of this small rectangle,

$$\mathbf{H}_0 = H_{x0}\mathbf{a}_x + H_{y0}\mathbf{a}_y + H_{z0}\mathbf{a}_z$$

The closed line integral of **H** about this path is then approximately the sum of the four values of $\mathbf{H} \cdot \Delta\mathbf{L}$ on each side. We choose the direction of traverse as 1-2-3-4-1, which corresponds to a current in the $\mathbf{a}_z$ direction, and the first contribution is therefore

$$(\mathbf{H} \cdot \Delta\mathbf{L})_{1-2} = H_{y,1-2}\Delta y$$

The value of H_y *on this section* of the path may be given in terms of the reference value H_{y0} at the center of the rectangle, the rate of change of H_y with x, and the distance $\Delta x/2$ from the center to the midpoint of side 1-2:

$$H_{y,1-2} \doteq H_{y0} + \frac{\partial H_y}{\partial x}\left(\tfrac{1}{2}\Delta x\right)$$

Thus

$$(\mathbf{H} \cdot \Delta\mathbf{L})_{1-2} \doteq \left(H_{y0} + \frac{1}{2}\frac{\partial H_y}{\partial x}\Delta x\right)\Delta y$$

Along the next section of the path we have

$$(\mathbf{H} \cdot \Delta\mathbf{L})_{2-3} \doteq H_{x,2-3}(-\Delta x) \doteq -\left(H_{x0} + \frac{1}{2}\frac{\partial H_x}{\partial y}\Delta y\right)\Delta x$$

Continuing for the remaining two segments and adding the results,

$$\oint \mathbf{H} \cdot d\mathbf{L} \doteq \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\Delta x \Delta y$$

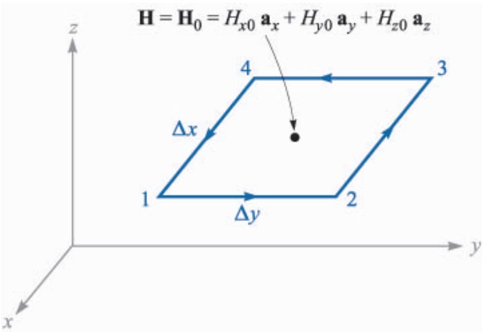

FIGURE 8.13
An incremental closed path in cartesian coordinates is selected for the application of Ampère's circuital law to determine the spatial rate of change of **H**.

By Ampère's circuital law, this result must be equal to the current enclosed by the path, or the current crossing any surface bounded by the path. If we assume a general current density **J**, the enclosed current is then $\Delta I \doteq J_z \Delta x \Delta y$, and

$$\oint \mathbf{H} \cdot d\mathbf{L} \doteq \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \Delta x \Delta y \doteq J_z \Delta x \Delta y$$

or

$$\frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta x \Delta y} \doteq \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \doteq J_z$$

As we cause the closed path to shrink, the above expression becomes more nearly exact, and in the limit we have the equality

$$\lim_{\Delta x, \Delta y \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta x \Delta y} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z \tag{18}$$

After beginning with Ampère's circuital law equating the closed line integral of **H** to the current enclosed, we have now arrived at a relationship involving the closed line integral of **H** *per unit area* enclosed and the current *per unit area* enclosed, or current density. We performed a similar analysis in passing from the integral form of Gauss's law, involving flux through a closed surface and charge enclosed, to the point form, relating flux through a closed surface *per unit volume* enclosed and charge *per unit volume* enclosed, or volume charge density. In each case a limit is necessary to produce an equality.

If we choose closed paths which are oriented perpendicularly to each of the remaining two coordinate axes, analogous processes lead to expressions for the y and z components of the current density,

$$\lim_{\Delta y, \Delta z \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta y \Delta z} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x \tag{19}$$

and

$$\lim_{\Delta z, \Delta x \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta z \Delta x} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y \tag{20}$$

Comparing (18), (19), and (20), we see that a component of the current density is given by the limit of the quotient of the closed line integral of **H** about a small path in a plane normal to that component and of the area enclosed as the path shrinks to zero. This limit has its counterpart in other fields of science and long ago received the name of *curl*. The curl of any vector is a vector, and any component of the curl is given by the limit of the quotient of the closed line integral of the vector about a small path in a plane normal to that component desired and the area enclosed, as the path shrinks to zero. It should be noted that the above definition of curl does not refer specifically to a particular coordinate system. The mathematical form of the definition is

$$(\text{curl } \mathbf{H})_N = \lim_{\Delta S_N \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta S_N} \tag{21}$$

where ΔS_N is the planar area enclosed by the closed line integral. The N subscript indicates that the component of the curl is that component which is *normal* to the surface enclosed by the closed path. It may represent any component in any coordinate system.

In cartesian coordinates the definition (21) shows that the x, y, and z components of the curl **H** are given by (18), (19), and (20), and therefore

$$\text{curl } \mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z \tag{22}$$

This result may be written in the form of a determinant,

$$\text{curl } \mathbf{H} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} \tag{23}$$

and may also be written in terms of the vector operator,

$$\text{curl } \mathbf{H} = \nabla \times \mathbf{H} \tag{24}$$

Equation (22) is the result of applying the definition (21) to the cartesian coordinate system. We obtained the z component of this expression by evaluating Ampère's circuital law about an incremental path of sides Δx and Δy, and we could have obtained the other two components just as easily by choosing the appropriate paths. Equation (23) is a neat method of storing the cartesian coordinate expression for curl; the form is symmetrical and easily remembered. Equation (24) is even more concise and leads to (22) upon applying the definitions of the cross product and vector operator.

The expressions for curl **H** in cylindrical and spherical coordinates are derived in Appendix A by applying the definition (21). Although they may be written in determinant form, as explained there, the determinants do not have one row of unit vectors on top and one row of components on the bottom, and they are not easily memorized. For this reason, the curl expansions in cylindrical and spherical coordinates which appear below and inside the back cover are usually referred to whenever necessary.

$$\nabla \times \mathbf{H} = \left(\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right)\mathbf{a}_\rho + \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}\right)\mathbf{a}_\phi + \left(\frac{1}{\rho}\frac{\partial(\rho H_\phi)}{\partial \rho} - \frac{1}{\rho}\frac{\partial H_\rho}{\partial \phi}\right)\mathbf{a}_z \quad \text{(cylindrical)} \tag{25}$$

$$\nabla \times \mathbf{H} = \frac{1}{r\sin\theta}\left(\frac{\partial(H_\phi \sin\theta)}{\partial \theta} - \frac{\partial H_\theta}{\partial \phi}\right)\mathbf{a}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial H_r}{\partial \phi} - \frac{\partial(rH_\phi)}{\partial r}\right)\mathbf{a}_\theta + \frac{1}{r}\left(\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta}\right)\mathbf{a}_\phi \quad \text{(spherical)} \tag{26}$$

Although we have described curl as a line integral per unit area, this does not provide everyone with a satisfactory physical picture of the nature of the curl operation, for the closed line integral itself requires physical interpretation. This integral was first met in the electrostatic field, where we saw that $\oint \mathbf{E} \cdot d\mathbf{L} = 0$. Inasmuch as the integral was zero, we did not belabor the physical picture. More recently we have discussed the closed line integral of $\mathbf{H}$, $\oint \mathbf{H} \cdot d\mathbf{L} = I$. Either of these closed line integrals is also known by the name of "circulation," a term obviously borrowed from the field of fluid dynamics.

The circulation of $\mathbf{H}$, or $\oint \mathbf{H} \cdot d\mathbf{L}$, is obtained by multiplying the component of $\mathbf{H}$ parallel to the specified closed path at each point along it by the differential path length and summing the results as the differential lengths approach zero and as their number becomes infinite. We do not require a vanishingly small path. Ampère's circuital law tells us that if $\mathbf{H}$ does possess circulation about a given path, then current passes through this path. In electrostatics we see that the circulation of $\mathbf{E}$ is zero about every path, a direct consequence of the fact that zero work is required to carry a charge around a closed path.

We may now describe curl as *circulation per unit area*. The closed path is vanishingly small, and curl is defined at a point. The curl of $\mathbf{E}$ must be zero, for the circulation is zero. The curl of $\mathbf{H}$ is not zero, however; the circulation of $\mathbf{H}$ per unit area is the current density by Ampère's circuital law [or (18), (19), and (20)].

Skilling[5] suggests the use of a very small paddle wheel as a "curl meter." Our vector quantity, then, must be thought of as capable of applying a force to each blade of the paddle wheel, the force being proportional to the component of the field normal to the surface of that blade. To test a field for curl we dip our paddle wheel into the field, with the axis of the paddle wheel lined up with the

[5] See the Suggested References at the end of the chapter.

direction of the component of curl desired, and note the action of the field on the paddle. No rotation means no curl; larger angular velocities mean greater values of the curl; a reversal in the direction of spin means a reversal in the sign of the curl. To find the direction of the vector curl and not merely to establish the presence of any particular component, we should place our paddle wheel in the field and hunt around for the orientation which produces the greatest torque. The direction of the curl is then along the axis of the paddle wheel, as given by the right-hand rule.

As an example, consider the flow of water in a river. Fig. 8.14*a* shows the longitudinal section of a wide river taken at the middle of the river. The water velocity is zero at the bottom and increases linearly as the surface is approached. A paddle wheel placed in the position shown, with its axis perpendicular to the paper, will turn in a clockwise direction, showing the presence of a component of curl in the direction of an inward normal to the surface of the page. If the velocity of water does not change as we go up- or downstream and also shows no variation as we go across the river (or even if it decreases in the same fashion toward either bank), then this component is the only component present at the center of the stream, and the curl of the water velocity has a direction into the page.

In Fig. 8.14*b* the streamlines of the magnetic field intensity about an infinitely long filamentary conductor are shown. The curl meter placed in this field of curved lines shows that a larger number of blades have a clockwise force exerted on them but that this force is in general smaller than the counterclockwise force exerted on the smaller number of blades closer to the wire. It seems possible that if the curvature of the streamlines is correct and also if the variation of the field strength is just right, the net torque on the paddle wheel may be zero. Actually, the paddle wheel does not rotate in this case, for since $\mathbf{H} = (I/2\pi\rho)\mathbf{a}_\phi$, we may substitute into (25) obtaining

$$\text{curl } \mathbf{H} = -\frac{\partial H_\phi}{\partial z}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial(\rho H_\phi)}{\partial \rho}\mathbf{a}_z = 0$$

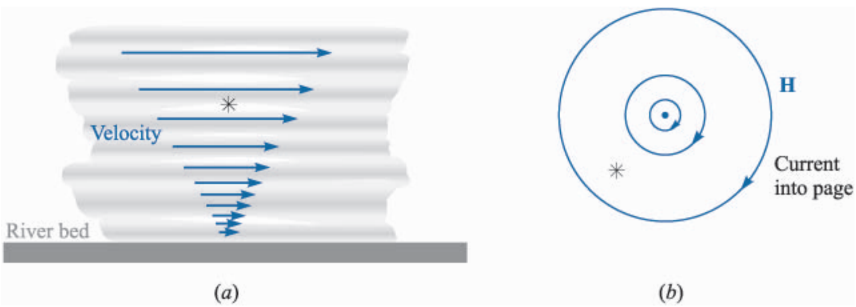

FIGURE 8.14
(*a*) The curl meter shows a component of the curl of the water velocity into the page. (*b*) The curl of the magnetic field intensity about an infinitely long filament is shown.

Example 8.2

As an example of the evaluation of curl $\mathbf{H}$ from the definition and of the evaluation of another line integral, let us suppose that $\mathbf{H} = 0.2z^2\mathbf{a}_x$ for $z > 0$, and $\mathbf{H} = 0$ elsewhere, as shown in Fig. 8.15. Calculate $\oint \mathbf{H} \cdot d\mathbf{L}$ about a square path with side d, centered at $(0, 0, z_1)$ in the $y = 0$ plane where $z_1 > 2d$.

Solution. We evaluate the line integral of $\mathbf{H}$ along the four segments, beginning at the top:

$$\oint \mathbf{H} \cdot d\mathbf{L} = 0.2(z_1 + \tfrac{1}{2}d)^2 d + 0 - 0.2(z_1 - \tfrac{1}{2}d)^2 d + 0$$
$$= 0.4 z_1 d^2$$

In the limit as the area approaches zero, we find

$$(\nabla \times \mathbf{H})_y = \lim_{d \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{d^2} = \lim_{d \to 0} \frac{0.4 z_1 d^2}{d^2} = 0.4 z_1$$

The other components are zero, so $\nabla \times \mathbf{H} = 0.4 z_1 \mathbf{a}_y$.

To evaluate the curl without trying to illustrate the definition or the evaluation of a line integral, we simply take the partial derivative indicated by (23):

$$\nabla \times \mathbf{H} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0.2z^2 & 0 & 0 \end{vmatrix} = \frac{\partial}{\partial z}(0.2z^2)\mathbf{a}_y = 0.4z\mathbf{a}_y$$

which checks with the result above when $z = z_1$.

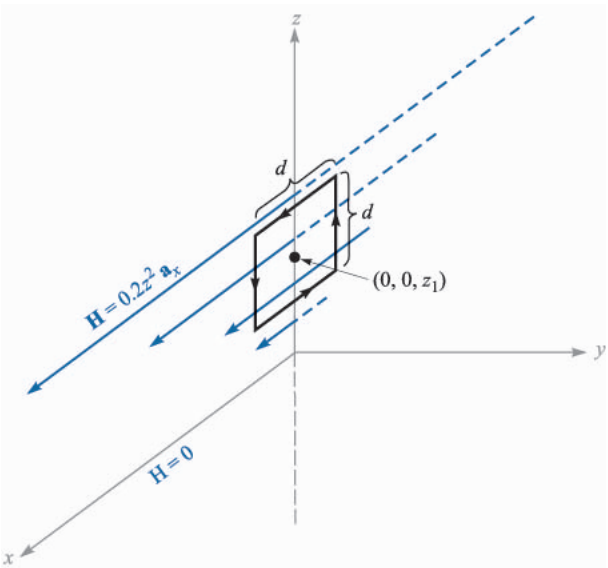

FIGURE 8.15
A square path of side d with its center on the z axis at $z = z_1$ is used to evaluate $\oint \mathbf{H} \cdot d\mathbf{L}$ and find curl $\mathbf{H}$.

Returning now to complete our original examination of the application of Ampère's circuital law to a differential-sized path, we may combine (18), (19), (20), (22), and (24),

$$\text{curl } \mathbf{H} = \nabla \times \mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z = \mathbf{J} \tag{27}$$

and write the *point form of Ampère's circuital law*,

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{28}$$

This is the second of Maxwell's four equations as they apply to non-time-varying conditions. We may also write the third of these equations at this time; it is the point form of $\oint \mathbf{E} \cdot d\mathbf{L} = 0$, or

$$\nabla \times \mathbf{E} = 0 \tag{29}$$

The fourth equation appears in Sec. 8.5.

✔ **D8.4.** (*a*) Evaluate the closed line integral of **H** about the rectangular path $P_1(2, 3, 4)$ to $P_2(4, 3, 4)$ to $P_3(4, 3, 1)$ to $P_4(2, 3, 1)$ to P_1, given $\mathbf{H} = 3z\mathbf{a}_x - 2x^3\mathbf{a}_z$ A/m. (*b*) Determine the quotient of the closed line integral and the area enclosed by the path as an approximation to $(\nabla \times \mathbf{H})_y$. (*c*) Determine $(\nabla \times \mathbf{H})_y$ at the center of the area.

Ans. 354 A; 59 A/m^2; 57 A/m^2

✔ **D8.5.** Calculate the value of the vector current density: (*a*) in cartesian coordinates at $P_A(2, 3, 4)$ if $\mathbf{H} = x^2 z\mathbf{a}_y - y^2 x\mathbf{a}_z$; (*b*) in cylindrical coordinates at $P_B(1.5, 90^\circ, 0.5)$ if $\mathbf{H} = \frac{2}{\rho}(\cos 0.2\phi)\mathbf{a}_\rho$; (*c*) in spherical coordinates at $P_C(2, 30^\circ, 20^\circ)$ if $\mathbf{H} = \frac{1}{\sin\theta}\mathbf{a}_\theta$.

Ans. $-16\mathbf{a}_x + 9\mathbf{a}_y + 16\mathbf{a}_z$ A/m^2; $0.0549\mathbf{a}_z$ A/m^2; $\mathbf{a}_\phi$ A/m^2

8.4 STOKES' THEOREM

Although the last section was devoted primarily to a discussion of the curl opeation, the contribution to the subject of magnetic fields should not be overlooked. From Ampère's circuital law we derived one of Maxwell's equations, $\nabla \times \mathbf{H} = \mathbf{J}$. This latter equation should be considered the point form of Ampère's circuital law and applies on a "per-unit-area" basis. In this section we shall again devote a major share of the material to the mathematical theorem known as Stokes' theorem, but in the process we shall show that we may obtain Ampère's circuital law from $\nabla \times \mathbf{H} = \mathbf{J}$. In other words, we are then prepared to